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A continuous dependence result for ultraparabolic equations in option pricing

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Abstract

We prove continuous dependence results for solution to the Cauchy problem related to degenerate parabolic equations arising in the valuation of financial derivatives. These results are crucial in some standard calibration procedure for recent stochastic volatility and interest rates models.

Keywords: Kolmogorov equation; Ultraparabolic equation; Option pricing; Continuous dependence

1. Introduction

In this paper we study the continuous dependence properties of solutions to parabolic equations whose coefficients are functions of a finite number of real parameters. We encountered this problem in [9,7] while examining a stochastic volatility model for pricing and hedging financial options. It is well known that in a Markovian setting the evaluation of derivative securities involves the study of the Cauchy problem related to some parabolic partial differential equation. It is the case of the standard heat equation in the classical Black and Scholes model [3]; while parabolic (possibly degenerate) equations, with variable coefficients, of the general form

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$$Lu := \sum_{i,j=1}^{N} a_{ij}\partial_{x_i x_j}u + \sum_{i=1}^{N} a_i\partial_{x_i}u + au - \partial_t u = 0, \quad (x,t) \in \mathbb{R}^{N+1}, \tag{1.1}$$

arise in more recent models. The coefficients of the equation depend on the so-called volatility structure which measures the uncertainty about future price movements of the assets underlying the option contract. As a matter of fact, the volatility is the key factor of a pricing model and its estimation is one of the main issues.

Given a set of quoted option prices $(u^*(x_i,t_i))_{i\in I}$, it is usual to calibrate a pricing model to the market by solving the inverse problem of finding those coefficients of L which make the model match (or at least approximate) the observed prices. The simplest way to do this is to parametrize the coefficients, that is to assume that $a_{ij} = a_{ij}(\cdot;\alpha), a_i = a_i(\cdot;\alpha), a = a(\cdot;\alpha)$ depend on a vector $\alpha = (\alpha_1,\dots,\alpha_p)$ of real numbers in a domain $\mathcal{A}$. Let us denote by $u(\cdot;\alpha)$ the solution to the Cauchy problem for (1.1) corresponding to α and with assigned initial condition: we look for that α which best fits the data by solving the nonlinear least squares problem

$$\min_{\alpha\in\mathcal{A}} \sum_{i\in I} \left|u(x_i,t_i;\alpha) - u^*(x_i,t_i)\right|^2 + \rho(\alpha), \tag{1.2}$$

where $\rho(\alpha)$ is some penalization term. In general this problem is not well posed (cf., for instance, [4]), however under suitable assumptions, standard numerical procedures based on the Newton method allow to select an approximate solution. This requires the computation, for $k = 1,\dots,p$, of the derivative $v_k = \frac{\partial u(\cdot;\alpha)}{\partial \alpha_k}$ that is solution, at least formally, of the equation

$$Lv_k = -\sum_{i,j=1}^{N} (\partial_{\alpha_k} a_{ij})\partial_{x_i x_j}u - \sum_{i=1}^{N} (\partial_{\alpha_k} a_i)\partial_{x_i}u - (\partial_{\alpha_k} a)u$$

obtained by differentiating (1.1) with respect to α_k. This fact is well known in the framework of standard uniformly parabolic equation where several results on the continuous dependence properties of solutions with respect to the parameters are available. On the contrary there are relevant kinds of financial derivatives like path-dependent options of Asian style (cf., for instance, [2] and [1]), or recent stochastic volatility models (cf. the Hobson and Rogers model [11] and path dependent volatility [10]), or some interest rates models (cf., for instance, [5] or [21] concerning the Markovian realization in the Heath–Jarrow–Morton framework) which involve hypoelliptic ultraparabolic equations for which such results, as far as we know, have not been proved. In the case of constant coefficients, the prototype of degenerate equations we are interested in is the following one:

$$\partial_{xx}u + x\partial_y u - \partial_t u = 0, \quad (x,y,t) \in \mathbb{R}^3. \tag{1.3}$$

Note that only one of the two space variables x, y appears in the second-order part of the equation. The aim of this paper is to prove continuous dependence results for solutions to the Cauchy problem for a general class of second-order linear equations with variable coefficients that includes (1.3). To this end we adapt and refine some techniques used in [8] where we proved existence and uniqueness results for the initial value problem.

The paper is organized as follows: in the next section we state the hypotheses and our main result, Theorem 2.3. In Section 3, we prove some estimates for the derivatives of the fundamental solution. Section 4 contains the proof of Theorem 2.3.

2. Main results and applications

We are concerned with second-order linear operators in the form

$$L^\alpha u(z) := \sum_{i,j=1}^{p_0} a_{ij}(z;\alpha)\partial_{x_i x_j} u(z) + \sum_{i=1}^{p_0} a_i(z;\alpha)\partial_{x_i} u(z) + a(z;\alpha)u(z)$$
$$+ \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u(z) - \partial_t u(z), \tag{2.1}$$

where $z = (x,t) \in \mathbb{R}^N \times \mathbb{R}$, $1 \leqslant p_0 \leqslant N$, $\alpha \in \mathcal{A}$. We assume the following hypotheses:

H1. The matrix $A(z;\alpha) = (a_{ij}(z;\alpha))_{i,j=1,\dots,p_0}$ is symmetric and uniformly positive definite in $\mathbb{R}^{p_0}$: there exists a positive constant μ such that

$$\frac{|\eta|^2}{\mu} \leqslant \sum_{i,j=1}^{p_0} a_{ij}(z;\alpha)\eta_i\eta_j \leqslant \mu|\eta|^2, \quad \eta \in \mathbb{R}^{p_0},\ z \in \mathbb{R}^{N+1},\ \alpha \in \mathcal{A}; \tag{2.2}$$

H2. The matrix $B := (b_{ij})$ has constant real entries and takes the following block form:

$$\begin{pmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{pmatrix}, \tag{2.3}$$

where B_j is a $p_{j-1} \times p_j$ matrix of rank p_j, with

$$p_0 \geqslant p_1 \geqslant \dots \geqslant p_r \geqslant 1, \quad p_0 + p_1 + \dots + p_r = N,$$

while the $*$-blocks are arbitrary;

H3. The coefficients a_{ij}, a_i and a are continuous functions. Moreover, a_{ij}, a_i and a are bounded and B-Hölder continuous of order $\delta \in]0,1[$ (in the sense of Definition 2.2) with respect to the variables (x,t), uniformly in α.

In order to briefly comment our hypotheses, we introduce some notations and recall some results for constant coefficients equations. Given a symmetric and positive definite matrix $\bar{A} = (a_{ij})_{i,j=1,\dots,p_0}$ with *constant* entries, we define the operator K in $\mathbb{R}^{N+1}$ as follows:

$$Ku := \sum_{i,j=1}^{p_0} a_{ij}\partial_{x_i x_j} u + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u - \partial_t u. \tag{2.4}$$

Then H2 is equivalent to any of the following properties (cf., for instance, [15]):

(i) K is hypoelliptic, i.e. every distributional solution of $Ku = f$ is a smooth classical solution whenever f is smooth;

(ii) if we set

$$E(t)=\exp\bigl(-tB^T\bigr),\qquad \mathcal{C}(t)=\int_0^t E(s)\begin{pmatrix}\bar{A} & 0\\ 0 & 0\end{pmatrix}E^T(s)\,ds, \tag{2.5}$$

then, for every $t>0$, the matrix $\mathcal{C}(t)$ is positive definite;

(iii) K satisfies the classical Hörmander condition:

$$\operatorname{rank}\operatorname{Lie}(\partial_{x_1},\dots,\partial_{x_{p_0}},Y)=N+1, \tag{2.6}$$

where $\operatorname{Lie}(\partial_{x_1},\dots,\partial_{x_{p_0}},Y)$ denotes the Lie algebra generated by the vector fields

$$\partial_{x_1},\dots,\partial_{x_{p_0}}$$

and

$$Y=\sum_{i,j=1}^N b_{ij}x_i\partial_{x_j}-\partial_t. \tag{2.7}$$

We also remark that if σ is a $N\times p_0$ matrix such that

$$\begin{pmatrix}\bar{A} & 0\\ 0 & 0\end{pmatrix}=\frac{1}{2}\sigma\sigma^T,$$

then K is the Kolmogorov operator related to the linear system of stochastic differential equations

$$dX_t=B^TX_t\,dt+\sigma\,dW_t, \tag{2.8}$$

where W denotes a standard p_0-dimensional Wiener process. It is well known that the solution X is a Gaussian process and that assumption H2 ensures that X has a transition density function which is the fundamental solution Γ_K of K (cf., for instance, [14, Chapter 5.6] or [12]). More explicitly we have

$$\Gamma_K(x,t,\xi,\tau)=\frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det\mathcal{C}(t-\tau)}}e^{-\frac{1}{4}\langle\mathcal{C}^{-1}(t-\tau)(x-E(t-\tau)\xi),x-E(t-\tau)\xi\rangle-(t-\tau)\operatorname{tr}B} \tag{2.9}$$

if $t>\tau$, and $\Gamma(x,t,\xi,\tau)=0$ if $t\leqslant\tau$.

Next we recall that K has remarkable invariance properties of with respect to a suitable Lie group structure on $\mathbb{R}^{N+1}$. These properties were first pointed out by Lanconelli and Polidoro in [15] who proved that K is invariant with respect to the left translation in the law defined by

$$(x,t)\circ(\xi,\tau)=\bigl(\xi+E(\tau)x,t+\tau\bigr),\quad (x,t),(\xi,\tau)\in\mathbb{R}^N\times\mathbb{R}, \tag{2.10}$$

where $E(\tau)$ is the matrix in (2.5). Moreover, if (and only if) all the $*$-block in (2.3) are null, then K is homogeneous of degree two with respect to the family of dilations $(D(\lambda))_{\lambda>0}$ defined by

$$D(\lambda):=\bigl(D_0(\lambda),\lambda^2\bigr):=\operatorname{diag}\bigl(\lambda I_{p_0},\lambda^3I_{p_1},\dots,\lambda^{2r+1}I_{p_r},\lambda^2\bigr), \tag{2.11}$$

where I_{p_j} denotes the $p_j\times p_j$ identity matrix: more precisely, we have

$$K\circ D(\lambda)=\lambda^2\bigl(D(\lambda)\circ K\bigr),\quad \lambda>0, \tag{2.12}$$

and Γ_K is $D(\lambda)$-homogeneous:

$$\Gamma_K\bigl(D(\lambda)z\bigr)=\lambda^{-Q}\Gamma_K(z),\quad z\in\mathbb{R}^{N+1}\setminus\{0\},\ \lambda>0,$$

where

$$Q = p_0 + 3p_1 + \cdots + (2r+1)p_r.$$

Since $\det(D(\lambda)) = \lambda^{Q+2}$, the number $Q+2$ is usually called the $D(\lambda)$-*homogeneous dimension* of $\mathbb{R}^{N+1}$.

In the case of Hölder continuous coefficients, Weber [23], Il'in [13] and Sonin [22] first used the parametrix method to construct a fundamental solution of (2.1). However unnecessary restrictive conditions on the regularity of the coefficients are assumed in these papers. General results under more natural assumptions were proved by Polidoro [20,18,19] in the homogeneous case (null $*$-blocks in (2.3)) and by Morbidelli [16] and the authors [8,17] in more general settings. In these papers the coefficients of the operator are supposed to be Hölder continuous functions with respect to the following $D(\lambda)$-homogeneous norm.

Definition 2.1. Given a constant matrix B of the form of (2.3) and $(D(\lambda))_{\lambda>0}$ defined as in (2.11), let $(q_j)_{j=1,\dots,N}$ be such that

$$D(\lambda) = \operatorname{diag}\big(\lambda^{q_1}, \lambda^{q_2}, \dots, \lambda^{q_N}, \lambda^2\big).$$

For every $z = (x,t) \in \mathbb{R}^N \times \mathbb{R}$, we set

$$|x|_B = \sum_{j=1}^{N} |x_j|^{\frac{1}{q_j}} \quad \text{and} \quad \|z\|_B = |x|_B + |t|^{\frac{1}{2}}. \tag{2.13}$$

Clearly $\|\cdot\|_B$ is a norm on $\mathbb{R}^{N+1}$ homogeneous of degree one with respect to the dilations $(D(\lambda))$.

Definition 2.2. A function F is B-Hölder continuous of order $\delta \in \,]0,1[$ on a domain Ω of $\mathbb{R}^{N+1}$ if

$$\big|F(z) - F(\zeta)\big| \leqslant C\big\|\zeta^{-1} \circ z\big\|^{\delta}, \quad z, \zeta \in \Omega, \tag{2.14}$$

for some positive constant C. In (2.14), ζ^{-1} denotes the inverse of ζ in the law "$\circ$" in (2.10).

Under assumptions H1–H3, in [8] we proved the existence of a fundamental solution Γ^α to L^α in (2.1) and some existence and uniqueness results for the related Cauchy problem

$$\begin{cases} L^\alpha u(x,t;\alpha) = f(x,t;\alpha), & (x,t) \in \mathbb{R}^N \times \,]0,T[, \\ u(\cdot,0;\alpha) = g(\cdot;\alpha). \end{cases} \tag{2.15}$$

More precisely, assume that f and g are continuous functions satisfying the growth conditions

$$\big|g(x;\alpha)\big| \leqslant c_1 e^{c_1|x|^2}, \quad x \in \mathbb{R}^N, \tag{2.16}$$

$$\big|f(x,t;\alpha)\big| \leqslant c_1 \frac{e^{c_1|x|^2}}{t^{1-\beta}}, \quad x \in \mathbb{R}^N,\ t \in \,]0,T[, \tag{2.17}$$

and, for every compact subset $M \subset \mathbb{R}^N$,

$$\big|g(x;\alpha) - g(x';\alpha)\big| \leqslant c_2|x - x'|_B^{\delta}, \tag{2.18}$$

$$\big|f(x,t;\alpha) - f(x',t;\alpha)\big| \leqslant c_2 \frac{|x - x'|_B^{\delta}}{t^{1-\beta}} \tag{2.19}$$

for some positive constants c_1, c_2 and $\beta > 0$, and for any $x, x' \in M$, $t \in]0, T[$, $\alpha \in \mathcal{A}$. Then (2.15) has a classical solution in the form

$$u(x,t;\alpha) = \int_{\mathbb{R}^N} \Gamma^\alpha(x,t,\xi,0) g(\xi;\alpha)\, d\xi - \int_0^t \int_{\mathbb{R}^N} \Gamma^\alpha(x,t,\xi,\tau) f(\xi,\tau;\alpha)\, d\xi\, d\tau \tag{2.20}$$

for $T > 0$ suitably small, only dependent on c_1. Next we state our main result.

Theorem 2.3. *Under hypotheses* H1–H3*, let $u(\cdot;\alpha)$ be the solution in* (2.20) *to problem* (2.15)*. Assume that*

(i) $\partial_{\alpha_k} g$ *is a continuous function satisfying* (2.16);
(ii) $\partial_{\alpha_k} f$ *is a continuous function satisfying* (2.17)–(2.19);
(iii) $\partial_{\alpha_k} a_{ij}$, $\partial_{\alpha_k} a_i$, $\partial_{\alpha_k} a$ *are continuous functions satisfying* (2.17)–(2.19) *with* $\beta > 1 - \delta/2$.

Then $u(x,t;\cdot) \in C^1(\mathcal{A})$ for every $(x,t) \in \mathbb{R}^N \times]0,T[$, and the partial derivative $\partial_{\alpha_k} u$ is solution to the Cauchy problem

$$\begin{cases} L^\alpha v = \partial_{\alpha_k} f - \sum_{i,j=1}^{p_0} (\partial_{\alpha_k} a_{ij}) \partial_{x_i x_j} u - \sum_{i=1}^{p_0} (\partial_{\alpha_k} a_i) \partial_{x_i} u - (\partial_{\alpha_k} a) u, & (x,t) \in \mathbb{R}^N \times]0,T[, \\ v(\cdot,0) = \partial_{\alpha_k} g \end{cases} \tag{2.21}$$

for any $k = 1, \dots, q$ and $\alpha \in \mathcal{A}$.

The proof of the theorem is postponed to Section 4 since it is based on some estimates of the fundamental solution (and its derivatives) which are provided in Section 3.

We close this section by briefly presenting an application of Theorem 2.3 to volatility modeling in finance. We recall that some extension of the standard local volatility has been recently proposed by Hobson and Rogers in [11], Foschi and one of the authors in [10]. In these papers the volatility is defined as a function of the whole trajectory of the underlying asset. Specifically, let us consider an average weight ψ that is a nonnegative, piecewise continuous and integrable function on $]-\infty, T]$. We assume that ψ is strictly positive in $[0,T]$ and we set

$$\Psi(t) = \int_{-\infty}^t \psi(s)\, ds.$$

Then we define the average process (or trend) as

$$Y_t = \frac{1}{\Psi(t)} \int_{-\infty}^t \psi(s) Z_s\, ds, \qquad t \in]0,T],$$

where $Z_t = \log(e^{-rt} S_t)$ denotes the log-discounted price process: the Hobson and Rogers model corresponds to the specification $\psi(t) = e^{\lambda t}$ for some positive parameter λ. Then by Itô formula we have

$$dY_t = \frac{\phi(t)}{\Phi(t)} (Z_t - Y_t)\, dt,$$

and assuming the following dynamic for the log-price

$$dZ_t = \mu(Z_t - Y_t)\,dt + \sigma(Z_t - Y_t)\,dW_t,$$

we obtain a system of stochastic differential equations of the form (2.8) where now σ is a non-constant function to be determined by calibration to market data. The idea is that, in case of large movements of the underlying asset far from its trend, the path-dependent volatility is designed to automatically increase its level in order to undertake market dynamics in a more natural way. The corresponding pricing differential equation is readily obtained by Itô formula

$$\frac{\sigma^2(z-y)}{2}(\partial_{zz} f - \partial_z f) + \frac{\phi(t)}{\Phi(t)}(z-y)\partial_y f + \partial_t f = 0, \quad (t,z,y) \in [0,T[\times \mathbb{R}^2. \tag{2.22}$$

In [9] and [10] a calibration procedure based on S&P500 option prices is derived: a NLLS problem of the form (1.2) is solved using the interior-point method described in [6]. This algorithm needs the first order derivatives $\partial_{\alpha_k} u$ which, by Theorem 2.3, are computed by solving a set of Cauchy problems of the form (2.21). For a detailed analysis of the calibration results and the performance of path dependent volatility compared with that of standard stochastic volatility models, we refer to [10]. More generally Theorem 2.3 applies to other models with dependence on the past like, for instance, Asian style options or interest rate models.

3. Estimates of the fundamental solution

In [8] we use the parametrix method to construct the fundamental solution of L^α under conditions H1–H3. Fixed $\alpha \in \mathbb{R}^q_+$ and $z_0 \in \mathbb{R}^{N+1}$, we define the "frozen" operator

$$K^\alpha_{z_0} = \sum_{i,j=1}^{p_0} a_{ij}(z_0;\alpha)\partial_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t \tag{3.1}$$

and denote by $\Gamma^\alpha_{z_0}$ its fundamental solution whose explicit expression is given in (2.9). We recall that a parametrix for L^α is defined by

$$Z^\alpha(z,\zeta) = \Gamma^\alpha_\zeta(z,\zeta), \tag{3.2}$$

and the parametrix method consists in looking for the fundamental solution Γ^α in the form

$$\Gamma^\alpha(z,\zeta) = Z^\alpha(z,\zeta) + \int_\tau^t \int_{\mathbb{R}^N} Z^\alpha(z,\omega)\phi^\alpha(\omega,\zeta)\,d\omega, \tag{3.3}$$

where ϕ^α is determined by imposing that $L^\alpha \Gamma^\alpha(z,\zeta) = 0$ for $z \neq \zeta$ and by successive approximations

$$\phi^\alpha(x,t,\xi,\tau) = \sum_{k=1}^{+\infty} LZ^{(\alpha)}_k(x,t,\xi,\eta), \quad x,\xi \in \mathbb{R}^N,\ 0 < \tau \leqslant t < T, \tag{3.4}$$

where

$$LZ^{(\alpha)}_1(x,t,\xi,\tau) = L^\alpha Z^\alpha(\xi,t,\xi t),$$

$$LZ^{(\alpha)}_{k+1}(x,t,\xi,\tau) = \int_\tau^t \int_{\mathbb{R}^N} L^\alpha Z^\alpha(x,t,y,s) LZ^{(\alpha)}_k(y,s,\xi,\tau)\,dy\,d\tau.$$

We state a preliminary

Lemma 3.1. *For every $\varepsilon > 0$ and $T > 0$, there exists a positive constant c_1 such that*

$$\left|\phi^\alpha(\xi_1,\tau,y,0)-\phi^\alpha(\xi_2,\tau,y,0)\right| \leqslant c_1 \frac{|\xi_1-\xi_2|_B^{\frac{\delta}{2}}}{\tau^{1-\frac{\delta}{4}}}\left(\Gamma_{K^\varepsilon}(\xi_1,\tau,y,0)+\Gamma_{K^\varepsilon}(\xi_2,\tau,y,0)\right), \tag{3.5}$$

$$\left|\phi^\alpha(\xi_1,\tau,y,0)\right| \leqslant c_1 \frac{\Gamma_{K^\varepsilon}(\xi_1,\tau,y,0)}{\tau^{1-\frac{\delta}{2}}}, \tag{3.6}$$

for every $\xi_1,\xi_2,y\in\mathbb{R}^N$, $t\in\,]0,T[$ and $\alpha\in\mathcal{A}$. Here Γ_{K^ε} is the fundamental solution of the operator

$$K^\varepsilon := (\mu+\varepsilon)\sum_{i,j=1}^{p_0}\partial_{x_i x_j}+\sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j}-\partial_t. \tag{3.7}$$

Estimate (3.6) is contained in Proposition 4.1 in [8], and (3.5) is a slightly different version of the estimate in Lemma 6.1 in [8] and can be proved analogously.

The parametrix method allows to obtain the following pointwise bounds of Γ^α and its derivatives (cf. [8, Proposition 3.5]): for every positive ε, T and polynomial function p, there exists a constant c that depends on T,μ,ε,p and B but not on α, such that, if we set $\eta=|D_0(\frac{1}{\sqrt{t-\tau}})(x-E(t-\tau)\xi)|$, then, for $i,j=1,\dots,p_0$, we have

$$\left|p(\eta)\right|\Gamma^\alpha(x,t,\xi,\tau)\leqslant c\,\Gamma_{K^\varepsilon}(x,t,\xi,\tau), \tag{3.8}$$

$$\left|p(\eta)\right|\left|\partial_{x_i}\Gamma^\alpha(x,t,\xi,\tau)\right|\leqslant c\frac{\Gamma_{K^\varepsilon}(x,t,\xi,t)}{\sqrt{t-\tau}}, \tag{3.9}$$

$$\left|p(\eta)\right|\left|\partial_{x_i x_j}\Gamma^\alpha(x,t,\xi,\tau)\right|\leqslant c\frac{\Gamma_{K^\varepsilon}(x,t,\xi,t)}{t-\tau}, \tag{3.10}$$

$$\left|p(\eta)\right|\left|Y\Gamma^\alpha(x,t,\xi,\tau)\right|\leqslant c\frac{\Gamma_{K^\varepsilon}(x,t,\xi,t)}{t-\tau}. \tag{3.11}$$

Here $Y\Gamma^\alpha$ denotes the Lie derivative with respect to the vector field Y defined in (2.7). As a further preliminary result, we also recall the reproduction property of Γ^α:

$$\Gamma^\alpha(x,t,\xi,\tau)=\int_{\mathbb{R}^N}\Gamma^\alpha(x,t,y,s)\Gamma^\alpha(y,s,\xi,\tau)\,dy,\quad \forall x,\xi\in\mathbb{R}^N,\ \tau<t,\ s\in\,]\tau,t[. \tag{3.12}$$

The main result of this section is the following

Theorem 3.2. *For every $T,\varepsilon>0$, there exists a positive constant c that depends on μ,B,T and ε but not on α, such that*

$$\left|\partial_{x_i}\Gamma^\alpha(x,t,y,0)-\partial_{x_i}\Gamma^\alpha(x',t,y,0)\right|\leqslant c\frac{|x-x'|_B^{\frac{\delta}{2}}}{t^{\frac12+\frac{\delta}{4}}}\Gamma_{K^\varepsilon}(x,t,y,0), \tag{3.13}$$

$$\left|\partial_{x_i x_j}\Gamma^\alpha(x,t,y,0)-\partial_{x_i x_j}\Gamma^\alpha(x',t,y,0)\right|\leqslant c\frac{|x-x'|_B^{\frac{\delta}{2}}}{t^{1+\frac{\delta}{4}}}\Gamma_{K^\varepsilon}(x,t,y,0) \tag{3.14}$$

for any $i, j = 1, \dots, p_0$, $t \in]0, T[$, $x, x', y \in \mathbb{R}^N$. *In the preceding estimates* δ *is the order of B-Hölder continuity of the coefficients of* L^α.

Proof. If $|x - x'|_B \geqslant \sqrt{t}$, then by (3.10) with $p \equiv 1$, there exists a positive constant c, only dependent on μ, B, T, ε such that

$$\left|\partial_{x_i x_j} \Gamma^\alpha(x, t, y, 0) - \partial_{x_i x_j} \Gamma^\alpha(x', t, y, 0)\right| \leqslant c \frac{\Gamma_{K^\varepsilon}(x, t, y, 0)}{t} \leqslant c \frac{|x - x'|_B^{\frac{\delta}{2}}}{t^{1+\frac{\delta}{4}}} \Gamma_{K^\varepsilon}(x, t, y, 0). \tag{3.15}$$

Next we consider the case $|x - x'|_B < \sqrt{t}$. By (3.3) we have

$$\partial_{x_i x_j} \Gamma^\alpha(x, t, y, 0) - \partial_{x_i x_j} \Gamma^\alpha(x, t, y, 0) = I_1 + I_2,$$

where

$$I_1 := \partial_{x_i x_j} Z^\alpha(x, t, y, 0) - \partial_{x_i x_j} Z^\alpha(x', t, y, 0),$$

$$I_2 := \int_0^t \int_{\mathbb{R}^N} \left(\partial_{x_i x_j} Z^\alpha(x, t, \xi, \tau) - \partial_{x_i x_j} Z^\alpha(x', t, \xi, \tau)\right) \phi^\alpha(\xi, \tau, y, 0)\, d\xi\, d\tau.$$

We only have to estimate I_2, since it is known (cf. formula (6.3) in [8]) that

$$|I_1| \leqslant c \frac{|x - x'|_B^{\frac{\delta}{2}}}{t^{1+\frac{\delta}{4}}} \Gamma_{K^\varepsilon}(x, t, y, 0) \tag{3.16}$$

for every $t \in]0, T[$, $x, x', y \in \mathbb{R}^N$, $|x - x'| \leqslant \sqrt{t}$. Then we split I_2 as the sum of J_1 and J_2 where

$$J_1 = \int_0^{\frac{t}{2}} \int_{\mathbb{R}^N} \left(\partial_{x_i x_j} Z^\alpha(x, t, \xi, \tau) - \partial_{x_i x_j} Z^\alpha(x', t, \xi, \tau)\right) \phi^\alpha(\xi, \tau, y, 0)\, d\xi\, d\tau,$$

$$J_2 = \int_{\frac{t}{2}}^t \int_{\mathbb{R}^N} \left(\partial_{x_i x_j} Z^\alpha(x, t, \xi, \tau) - \partial_{x_i x_j} Z^\alpha(x', t, \xi, \tau)\right) \phi^\alpha(\xi, \tau, y, 0)\, d\xi\, d\tau.$$

By (3.16) and (3.6), we have

$$|J_1| \leqslant c \int_0^{\frac{t}{2}} \int_{\mathbb{R}^N} \frac{|x - x'|_B^{\frac{\delta}{2}}}{(t - \tau)^{1+\frac{\delta}{4}}} \Gamma_{K^\varepsilon}(x, t, \xi, \tau) \frac{\Gamma_{K^\varepsilon}(\xi, \tau, y, 0)}{\tau^{1-\frac{\delta}{2}}}\, d\xi\, d\tau,$$

so that, by the reproduction property (3.12), we get

$$|J_1| \leqslant c \frac{|x - x'|_B^{\frac{\delta}{2}}}{t^{1+\frac{\delta}{4}}} \Gamma_{K^\varepsilon}(x, t, y, 0) \int_0^{\frac{t}{2}} \frac{1}{\tau^{1-\frac{\delta}{2}}}\, d\tau = c' \frac{|x - x'|_B^{\frac{\delta}{2}}}{t^{1-\frac{\delta}{4}}} \Gamma_{K^\varepsilon}(x, t, y, 0). \tag{3.17}$$

On the other hand we proceed as in the proof of Proposition 5.3 in [8] in order to estimate J_2. For $\tau \in]0,t[$ and $w \in \mathbb{R}^N$, we set $\omega = (w,\tau)$ and since

$$\int_{\mathbb{R}^N} \partial_{x_i x_j} \Gamma_\omega^\alpha(x,t,\xi,\tau)\,d\xi = 0,$$

we have $J_2 = \int_{\frac{t}{2}}^t (K_1(\tau) + K_2(\tau))\,d\tau$ where

$$\begin{aligned} K_1(\tau) &:= \int_{\mathbb{R}^N} \big(\partial_{x_i x_j} Z^\alpha(x,t,\xi,\tau) - \partial_{x_i x_j} Z^\alpha(x',t,\xi,\tau)\big) \\ &\quad\times \big(\phi^\alpha(\xi,\tau,y,0) - \phi^\alpha(w,\tau,y,0)\big)\,d\xi, \\ K_2(\tau) &:= \phi^\alpha(w,\tau,y,0) \int_{\mathbb{R}^N} \big(\partial_{x_i x_j} Z^\alpha(x,t,\xi,\tau) - \partial_{x_i x_j} Z^\alpha(x',t,\xi,\tau)\big) \\ &\quad - \big(\partial_{x_i x_j} \Gamma_\omega^\alpha(x,t,\xi,\tau) - \partial_{x_i x_j} \Gamma_\omega^\alpha(x',t,\xi,\tau)\big)\,d\xi. \end{aligned}$$

We put $w = E(\tau - t)x$ so that by (3.16) and (3.5), we infer

$$\begin{aligned} |K_1| &\leqslant c \int_{\mathbb{R}^N} \frac{|x-x'|_B^{\frac{\delta}{2}}}{(t-\tau)^{1+\frac{\delta}{4}}} \Gamma_{K^{\frac{\varepsilon}{2}}}(x,t,\xi,\tau) \frac{|\xi - w|_B^\delta}{\tau} \big(\Gamma_{K^\varepsilon}(\xi,\tau,y,0) + \Gamma_{K^\varepsilon}(w,\tau,y,0)\big)\,d\xi \\ &\leqslant c'|x-x'|_B^{\frac{\delta}{2}} \int_{\mathbb{R}^N} \frac{\Gamma_{K^{\frac{\varepsilon}{2}}}(x,t,\xi,\tau)}{(t-\tau)^{1-\frac{\delta}{4}}} \frac{|D_0(\frac{1}{\sqrt{t-\tau}})(x - E(t-\tau)\xi)|_B^\delta}{\tau} \\ &\quad \times \big(\Gamma_{K^\varepsilon}(\xi,\tau,y,0) + \Gamma_{K^\varepsilon}(w,\tau,y,0)\big)\,d\xi \end{aligned}$$

(since by (3.10), we have $\Gamma_{K^{\frac{\varepsilon}{2}}}(x,t,\xi,\tau)|D_0(\frac{1}{\sqrt{t-\tau}})(x-E(t-\tau)\xi)|_B^\delta \leqslant c\Gamma_{K^\varepsilon}(x,t,\xi,\tau)$)

$$\leqslant c'' \frac{|x-x'|_B^{\frac{\delta}{2}}}{(t-\tau)^{1-\frac{\delta}{4}}\tau} \int_{\mathbb{R}^N} \Gamma_{K^\varepsilon}(x,t,\xi,\tau)\big(\Gamma_{K^\varepsilon}(\xi,\tau,y,0) + \Gamma_{K^\varepsilon}(w,\tau,y,0)\big)\,d\xi$$

(by the reproduction property (3.12) and since $\int_{\mathbb{R}^N} \Gamma_{K^\varepsilon}(x,t,\xi,\tau)\,d\xi = 1$ for $t > \tau$)

$$= c'' \frac{|x-x'|_B^{\frac{\delta}{2}}}{(t-\tau)^{1-\frac{\delta}{4}}\tau} \big(\Gamma_{K^\varepsilon}(x,t,y,0) + \Gamma_{K^\varepsilon}(w,\tau,y,0)\big). \tag{3.18}$$

Now we remark that by the explicit expression (2.9) of Γ_{K^ε} and since the quadratic form associated with $\mathcal{C}(t)$ is a monotone increasing function of t, there exists a positive constant c such that

$$\Gamma_{K^\varepsilon}\big(E(\tau - t)x, \tau, y, 0\big) \leqslant c\Gamma_{K^{2\varepsilon}}(x,t,y,0), \quad \forall t \in]0,T[,\ \tau \in]t/2,t[,\ x,y, \in \mathbb{R}^N. \tag{3.19}$$

Therefore we have

$$\int_{\frac{t}{2}}^{t} |K_1|\,d\tau \leqslant c\frac{|x-x'|_B^{\frac{\delta}{2}}}{t}\Gamma_{K^{2\varepsilon}}(x,t,y,0)\int_{\frac{t}{2}}^{t}\frac{1}{(t-\tau)^{1-\frac{\delta}{4}}}\,d\tau$$
$$\leqslant c'\frac{|x-x'|_B^{\frac{\delta}{2}}}{t^{1-\frac{\delta}{4}}}\Gamma_{K^{2\varepsilon}}(x,t,y,0). \tag{3.20}$$

We now recall the notation $\omega=(w,\tau)$ and consider the term K_2. By Lemma 5.2 in [8], we have that, for every positive ε and T, there exists a positive constant c such that

$$\left|\partial_{x_ix_j}Z^\alpha(x,t,\xi,\tau)-\partial_{x_ix_j}Z^\alpha(x',t,\xi,\tau)-\partial_{x_ix_j}\Gamma_\omega^\alpha(x,t,\xi,\tau)-\partial_{x_ix_j}\Gamma_\omega^\alpha(x',t,\xi,\tau)\right|$$
$$\leqslant c|x-x'|_B^{\frac{\delta}{2}}\frac{\|(\xi,\tau)^{-1}\circ\omega\|_B^\delta}{(t-\tau)^{1+\frac{\delta}{4}}}\Gamma_{K^{\frac{\varepsilon}{2}}}(x,t,\xi,\tau)$$

for any $i,j=1,\dots,p_0$, $x,x',w\in\mathbb{R}^N$ and $0\leqslant t-\tau\leqslant T$. By the previous inequality and by (3.6), setting $w=E(\tau-t)x$ as before, we get

$$|K_2|\leqslant c\frac{\Gamma_{K^\varepsilon}(E(\tau-t)x,\tau,y,0)}{\tau^{1-\frac{\delta}{2}}}|x-x'|_B^{\frac{\delta}{2}}$$
$$\times\int_{\mathbb{R}^N}\frac{|D_0(\frac{1}{\sqrt{t-\tau}})(x-E(t-\tau)\xi)|_B^\delta}{(t-\tau)^{1-\frac{\delta}{4}}}\Gamma_{K^{\frac{\varepsilon}{2}}}(x,t,\xi,\tau)\,d\xi$$

(by using again (3.8) and (3.19))

$$\leqslant c\frac{\Gamma_{K^\varepsilon}(x,\tau,y,0)}{\tau^{1-\frac{\delta}{2}}}\frac{|x-x'|_B^{\frac{\delta}{2}}}{(t-\tau)^{1-\frac{\delta}{4}}}. \tag{3.21}$$

Therefore we finally have

$$\int_{\frac{t}{2}}^{t}|K_2|\,d\tau\leqslant c\frac{|x-x'|_B^{\frac{\delta}{2}}}{t^{1-\frac{\delta}{2}}}\Gamma_{K^\varepsilon}(x,t,y,0), \tag{3.22}$$

which concludes the proof. □

Now let us consider the solution $u(\cdot;\alpha)$ in (2.20) to the Cauchy problem (2.15). We aim to study the asymptotic behavior of $\partial_{x_i}u(\cdot,t;\alpha)$ and $\partial_{x_ix_j}u(\cdot,t;\alpha)$ as $t\to0^+$. We first recall the following identities proved in [8]: for every $i,j=1,\dots,p_0$, we have

$$\partial_{x_i}u(x,t;\alpha)=\int_{\mathbb{R}^N}\partial_{x_i}\Gamma^\alpha(x,t,y,0)g(y;\alpha)\,dy$$
$$-\int_0^t\int_{\mathbb{R}^N}\partial_{x_i}\Gamma^\alpha(x,t,y,s)f(y,s;\alpha)\,dy\,ds, \tag{3.23}$$

$$\partial_{x_i x_j} u(x,t;\alpha) = \int_{\mathbb{R}^N} \partial_{x_i x_j} \Gamma^\alpha(x,t,y,0) g(y;\alpha)\,dy \\ - \int_0^t \int_{\mathbb{R}^N} \partial_{x_i x_j} \Gamma^\alpha(x,t,y,s) f(y,s;\alpha)\,dy\,ds. \tag{3.24}$$

These formulas were proved in [8, Theorem 1.4, Propositions 5.1, 5.3 and 5.4], for f, g satisfying the usual conditions (2.16), (2.16) and (2.19) in the case $\beta = 1$: nevertheless the result is still valid in the general case $\beta \in]0,1]$, the proof being analogous.

Proposition 3.3. *Consider the Cauchy problem* (2.15) *under conditions* (2.16)–(2.19). *Then there exists $T > 0$ and a positive constant c, dependent on μ, B and T but not on α, such that*

$$\left|\partial_{x_i} u(x,t;\alpha)\right| \leqslant c \frac{e^{c|x|^2}}{t^{\frac{1}{2}-\frac{\delta}{2}}}, \tag{3.25}$$

$$\left|\partial_{x_i x_j} u(x,t;\alpha)\right| + \left|Yu(x,t;\alpha)\right| \leqslant c \frac{e^{c|x|^2}}{t^{1-\frac{\delta}{2}}} \tag{3.26}$$

for every $(x,t) \in \mathbb{R}^N \times]0,T[$, $i,j = 1,\dots,p_0$.

Proof. We only sketch the proof of the estimate of $\partial_{x_i x_j} u(x,t;\alpha)$ in the homogeneous case with null f, a and a_i, $i = 1,\dots,p_0$: in general the thesis follows by a similar argument by using the representation formula (3.3) of Γ^α in terms of the parametrix. The idea is that, since

$$\int_{\mathbb{R}^N} \Gamma^\alpha(x,t,\xi,\tau)\,d\xi = 1, \quad 0 \leqslant \tau < t \leqslant T,$$

by (3.24) we have

$$0 = \partial_{x_i x_j} \int_{\mathbb{R}^N} \Gamma^\alpha(x,t,\xi,0)\,d\xi = \int_{\mathbb{R}^N} \partial_{x_i x_j} \Gamma^\alpha(x,t,\xi,0)\,d\xi,$$

so that

$$\partial_{x_i x_j} u(x,t;\alpha) = I_1(x,t;\alpha) - I_2(x,t;\alpha),$$

where

$$I_1(x,t;\alpha) = \int_{\mathbb{R}^N} \partial_{x_i x_j} \Gamma^\alpha(x,t,\xi,0)\big(g(\xi;\alpha) - g\big(E(-t)x;\alpha\big)\big)\,d\xi,$$

$$I_2(x,t;\alpha) = \int_0^t \int_{\mathbb{R}^N} \partial_{x_i x_j} \Gamma^\alpha(x,t,\xi,\tau)\big(f(\xi,\tau;\alpha) - f\big(E(\tau-t)x,\tau;\alpha\big)\big)\,d\xi\,d\tau.$$

Then I_1, I_2 can be estimated proceeding as in the proof of Theorem 2.3, by using the Gaussian upper bounds (3.8)–(3.11) and the assumptions on f and g. □

Proposition 3.4. *Under the hypotheses of Proposition* 3.3, *for any compact subset M of $\mathbb{R}^N$, there exits a positive constant c, dependent on M, B, T and μ but not on α, such that*

$$\left|\partial_{x_i} u(x,t;\alpha) - \partial_{x_i} u(x',t;\alpha)\right| \leqslant c \frac{|x-x'|_B^{\frac{\delta}{2}}}{t^{\frac{1}{2}-\frac{\delta}{4}}}, \tag{3.27}$$

$$\left|\partial_{x_i x_j} u(x,t;\alpha) - \partial_{x_i x_j} u(x',t;\alpha)\right| \leqslant c \frac{|x-x'|_B^{\frac{\delta}{2}}}{t^{1-\frac{\delta}{4}}} \tag{3.28}$$

for every $x, x' \in M$ and $t \in \,]0,T[$.

Proof. The thesis is a straightforward consequence of the estimates of Theorem 3.2 since, assuming for simplicity $f = 0$, we have

$$\begin{aligned}&\partial_{x_i x_j} u(x,t;\alpha) - \partial_{x_i x_j} u(x',t;\alpha)\\ &\quad= \int\limits_{\mathbb{R}^N} \bigl(\partial_{x_i x_j} \Gamma^\alpha(x,t,y,0) - \partial_{x_i x_j} \Gamma^\alpha(x',t,y,0)\bigr) g(y;\alpha)\, dy. \qquad \square\end{aligned}$$

4. Proof of Theorem 2.3

In this section we prove Theorem 2.3. We begin with a preliminary

Lemma 4.1. *Under the hypotheses of Proposition* 3.3*, $u(\cdot;\alpha)$, $\partial_{x_h} u(\cdot;\alpha)$, $\partial_{x_h x_k} u(\cdot;\alpha)$, for $h,k = 1,\dots,p_0$, and $Yu(\cdot;\alpha)$ are continuous functions of the variable α.*

Proof. We only consider $\partial_{x_h x_k} u(\cdot;\alpha)$, since the proof of the continuity of $u(\cdot;\alpha)$, $\partial_{x_h} u(\cdot;\alpha)$ and $Yu(\cdot;\alpha)$ is analogous. Moreover, for simplicity, we only consider the case $a_i = a \equiv 0$.

We have

$$\begin{cases} L^{\alpha_0}(u(x,t;\alpha) - u(x,t;\alpha_0)) = F(x,t;\alpha), & (x,t) \in \mathbb{R}^N \times \,]0,T[,\\ u(x,0;\alpha) - u(x,0;\alpha_0) = g(x;\alpha) - g(x;\alpha_0), & x \in \mathbb{R}^N,\end{cases}$$

where

$$F(x,t;\alpha) := f(x,t;\alpha) - f(x,t;\alpha_0) + \sum_{i,j=1}^{p_0} \bigl(a_{ij}(x,t;\alpha_0) - a_{ij}(x,t;\alpha)\bigr)\partial_{x_i x_j} u(x,t;\alpha).$$

Since a_{ij} are bounded uniformly in α and by (2.17), (3.26), we have

$$\left|F(x,t;\alpha)\right| \leqslant c \frac{e^{c|x|^2}}{t^{1-\gamma}}, \quad (x,t) \in \mathbb{R}^N \times \,]0,T[, \tag{4.1}$$

for some constant c independent of α, where $\gamma = \min\{\beta, \delta/2\}$. Moreover, since $a_{ij}(\cdot;\alpha)$ are B-Hölder continuous of order δ uniformly in α and by Proposition 3.4, we have that for every M compact subset of $\mathbb{R}^N$ there exists a positive constant c such that

$$\left|F(x,t;\alpha) - F(x',t;\alpha)\right| \leqslant c \frac{|x-x'|^{\frac{\delta}{2}}}{t^{1-\gamma'}}, \quad \forall x, x' \in M,\ t \in \,]0,T[, \tag{4.2}$$

where $\gamma' = \min\{\beta, \delta/4\}$. Then, by (2.20) we have the following representation formula:

$$\partial_{x_h x_k}\bigl(u(x,t;\alpha) - u(x,t;\alpha_0)\bigr) = I_1(x,t;\alpha,\alpha_0) + I_2(x,t;\alpha,\alpha_0),$$

where, for $h,k = 1,\dots,p_0$,

$$I_1(x,t;\alpha,\alpha_0)=\int_{\mathbb{R}^N}\partial_{x_hx_k}\Gamma^{\alpha_0}(x,t,\xi,0)\big(g(\xi;\alpha)-g(\xi;\alpha_0)\big)\,d\xi,$$

$$I_2(x,t;\alpha,\alpha_0)=-\int_0^t\int_{\mathbb{R}^N}\partial_{x_hx_k}\Gamma^{\alpha_0}(x,t,\xi,\tau)F(\xi,\tau;\alpha)\,d\xi\,d\tau.$$

By (3.10) and (2.16), we have

$$\big|\partial_{x_hx_k}\Gamma^{\alpha_0}(x,t,\xi,0)\big|\leqslant c\frac{\Gamma_{K^\varepsilon}(x,t,\xi,0)e^{c|\xi|^2}}{t}\in L^1\big(\mathbb{R}^N\big),$$

provided that T is suitably small, with c independent of α. Therefore by the dominated convergence theorem we have

$$\lim_{\alpha\to\alpha_0}I_1(x,t;\alpha,\alpha_0)=0,\quad (x,t)\in\mathbb{R}^N\times\,]0,T[.$$

Fixed $\theta>0$, we set

$$I_2^\theta(x,t;\alpha,\alpha_0)=-\int_0^{t-\theta}\int_{\mathbb{R}^N}\partial_{x_hx_k}\Gamma^{\alpha_0}(x,t,\xi,\tau)F(\xi,\tau;\alpha)\,d\xi\,d\tau. \tag{4.3}$$

Since estimates (4.1), (4.2) and (3.10) hold uniformly on α, a standard argument shows

$$\lim_{\theta\to0^+}I_2^\theta(x,t;\alpha,\alpha_0)=I_2(x,t;\alpha,\alpha_0), \tag{4.4}$$

uniformly in α. On the other hand, by (4.1) and (3.10), there exists a positive constant c, independent of α, such that

$$\big|\partial_{x_hx_k}\Gamma^{\alpha_0}(x,t,\xi,\tau)F(\xi,\tau;\alpha)\big|\leqslant c\frac{\Gamma_{K^\varepsilon}(x,t,\xi,\tau)e^{c|\xi|^2}}{\theta\tau^{1-\gamma}}\in L^1\big(\mathbb{R}^N\times\,]0,t-\theta[\big), \tag{4.5}$$

provided that T is suitably small. Since a_{ij} are continuous functions, by the dominated convergence theorem, we have

$$\lim_{\alpha\to\alpha_0}I_2^\theta(x,t;\alpha,\alpha_0)=0,\quad (x,t)\in\mathbb{R}^N\times\,]0,T[,$$

and we infer

$$\lim_{\alpha\to\alpha_0}I_2(x,t;\alpha,\alpha_0)=0,\quad (x,t)\in\mathbb{R}^N\times\,]0,T[.\qquad\square$$

Proof of Theorem 2.3. We only consider the case $q=1$ and $a_i=a\equiv0$. We have

$$\frac{u(x,t;\alpha)-u(x,t;\alpha_0)}{\alpha-\alpha_0}=(I_1-I_2+I_3)(x,t;\alpha,\alpha_0),$$

where

$$I_1(x,t;\alpha,\alpha_0)=\sum_{i,j=1}^{p_0}\int_0^t\int_{\mathbb{R}^N}\Gamma^{\alpha_0}(x,t,\xi,\tau)\frac{a_{ij}(\xi,\tau;\alpha)-a_{ij}(\xi,\tau;\alpha_0)}{\alpha-\alpha_0}\partial_{\xi_i\xi_j}u(\xi,\tau;\alpha)\,d\xi\,d\tau,$$

$$I_2(x,t;\alpha,\alpha_0)=\int_0^t\int_{\mathbb{R}^N}\Gamma^{\alpha_0}(x,t,\xi,\tau)\frac{f(\xi,\tau;\alpha)-f(\xi,\tau;\alpha_0)}{\alpha-\alpha_0}\,d\xi\,d\tau,$$

$$I_3(x,t;\alpha,\alpha_0) = \int_{\mathbb{R}^N} \Gamma^{\alpha_0}(x,t,\xi,0)\frac{g(\xi;\alpha)-g(\xi;\alpha_0)}{\alpha-\alpha_0}\,d\xi.$$

By (3.26), the integral kernel in I_1 is estimated by

$$\left|\Gamma^{\alpha_0}(x,t,\xi,\tau)\frac{a_{ij}(\xi,\tau;\alpha)-a_{ij}(\xi,\tau;\alpha_0)}{\alpha-\alpha_0}\partial_{\xi_i\xi_j}u(\xi,\tau;\alpha)\right|$$
$$\leqslant c\Gamma^{\alpha_0}(x,t,\xi,\tau)\frac{e^{c|\xi|^2}}{\tau^{1-(\beta-1+\delta/2)}} \in L^1\big(\mathbb{R}^N\times\,]0,T[\big),$$

since, by assumption, $\beta > 1-\delta/2$. Therefore we use the continuity result in Lemma 4.1 and the dominated convergence theorem to get

$$\lim_{\alpha\to\alpha_0} I_1(x,t;\alpha,\alpha_0) = \sum_{i,j=1}^{p_0}\int_0^t\int_{\mathbb{R}^N}\Gamma^{\alpha_0}(x,t,\xi,\tau)\frac{da_{ij}(\xi,\tau;\alpha_0)}{d\alpha}\partial_{\xi_i\xi_j}u(\xi,\tau;\alpha_0)\,d\xi\,d\tau.$$

Terms I_2 and I_3 can be handled analogously to conclude the proof. □

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